Gromov-Witten theory and Donaldson-Thomas theory, II

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1 Introduction

1.1 Overview

The Gromov-Witten theory of a 3-fold X is defined via integrals over the moduli space of stable maps. The Donaldson-Thomas theory of X is defined via integrals over the moduli space of ideal sheaves. In [14], a GW/DT correspondence equating the two theories was proposed, and the Calabi-Yau case was presented. We discuss here the GW/DT correspondence for general 3-folds.

Let X be a nonsingular, projective 3-fold. Insertions in the Gromov-Witten theory of X are determined by primary and descendent fields. Insertions in the Donaldson-Thomas theory of X are naturally obtained from the Chern classes of universal sheaves. We conjecture a GW/DT correspondence for 3-folds relating these two sets of insertions.

Let $S \subset X$ be a nonsingular surface. The Gromov-Witten theory of X relative to S has been defined in [4, 9, 10, 12]. The relative constraints are determined by partitions weighted by cohomology classes of S. A relative Donaldson-Thomas theory has been defined by J. Li [13]. The relative constraints are determined by cohomology classes of the Hilbert scheme of points of S. We propose a GW/DT correspondence in the relative case relating the Gromov-Witten constraints to the Donaldson-Thomas constraints via Nakajima's basis of the cohomology of the Hilbert scheme of points.

In the last Section of the paper, independent of the conjectural framework, we study the Donaldson-Thomas theory in degree 0 using localization and

relative geometry. We derive a formula for the equivariant vertex measure in the degree 0 case and prove Conjecture $1'$ of [14] in the toric case. A degree 0 relative formula is also proven.

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2 The GW/DT correspondence for 3-folds

2.1 GW theory

Gromov-Witten theory is defined via integration over the moduli space of stable maps. Let X be a nonsingular, projective 3-fold. Let $\overline{M}_{g,r}(X,\beta)$ denote the moduli space of r-pointed stable maps from connected, genus g curves to X representing the class $\beta \in H_2(X,\mathbb{Z})$. Let

$$\mathrm{ev}_i : \overline{M}_{g,r}(X,\beta) \to X,$$

$$L_i \to \overline{M}_{g,r}(X,\beta)$$

denote the evaluation maps and cotangent lines bundles associated to the marked points. Let $\gamma_1, \ldots, \gamma_m$ be a basis of $H^*(X,\mathbb{Q})$, and let

$$\psi_i = c_1(L_i) \in \overline{M}_{g,n}(X,\beta).$$

The *descendent* fields, denoted by $\tau_k(\gamma_i)$, correspond to the classes $\psi_i^k \mathrm{ev}_i^*(\gamma_j)$ on the moduli space of maps. Let

$$\langle \tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r}) \rangle_{g,\beta} = \int_{[\overline{M}_{g,r}(X,\beta)]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_{l_i})$$

denote the descendent Gromov-Witten invariants. Foundational aspects of the theory are treated, for example, in [1, 2, 11].

Let C be a possibly disconnected curve with at worst nodal singularities. The genus of C is defined by $1-\chi(\mathcal{O}_C)$. Let $\overline{M}'_{g,r}(X,\beta)$ denote the moduli space of maps with possibly disconnected domain curves C of genus g with *no* collapsed connected components. The latter condition requires each connected component of C to represent a nontrivial class in $H_2(X,\mathbb{Z})$. In particular, C must represent a *nonzero* class β.

The descendent invariants are defined in the disconnected case by

$$\langle \tau_{k_1}(\gamma_{l_1})\cdots\tau_{k_r}(\gamma_{l_r})\rangle'_{g,\beta} = \int_{[\overline{M}'_{g,r}(X,\beta)]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_{l_i}).$$

Define the following generating function,

$$\mathsf{Z}'_{GW}\Big(X;u \mid \prod_{i=1}^{r}\tau_{k_i}(\gamma_{l_i})\Big)_\beta = \sum_{g\in\mathbb{Z}} \langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_{l_i})\rangle'_{g,\beta}\ u^{2g-2}. \tag{1}$$

Since the domain components must map nontrivially, an elementary argument shows the genus g in the sum (1) is bounded from below. The descendent insertions in (1) should match the (genus independent) virtual dimension,

$$\dim\ [\overline{M}'_{g,r}(X,\beta)]^{vir} = \int_\beta c_1(T_X) + r.$$

Following the terminology of [14], we view (1) as a *reduced* partition function.

2.2 DT theory

Donaldson-Thomas theory is defined via integration over the moduli space of ideal sheaves. Let X be a nonsingular, projective 3-fold. An *ideal sheaf* is a torsion-free sheaf of rank 1 with trivial determinant. Each ideal sheaf $\mathcal{I}$ injects into its double dual,

$$0 \to \mathcal{I} \to \mathcal{I}^{\vee\vee}.$$

As $\mathcal{I}^{\vee\vee}$ is reflexive of rank 1 with trivial determinant,

$$\mathcal{I}^{\vee\vee} \cong \mathcal{O}_X,$$

see [17]. Each ideal sheaf $\mathcal{I}$ determines a subscheme $Y \subset X$,

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

The maximal dimensional components of Y (weighted by their intrinsic multiplicities) determine an element,

$$[Y] \in H_*(X, \mathbb{Z}).$$

Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves $\mathcal{I}$ satisfying

$$\chi(\mathcal{O}_Y) = n,$$

and

$$[Y] = \beta \in H_2(X, \mathbb{Z}).$$

Here, χ denotes the holomorphic Euler characteristic.

The Donaldson-Thomas invariant is defined via integration against virtual class,

$$[I_n(X, \beta)]^{vir}.$$

Foundational aspects of the theory are treated in [15, 20].

Lemma 1. *The virtual dimension of $I_n(X, \beta)$ equals $\int_\beta c_1(T_X)$.*

Proof. The virtual dimension, obtained from the obstruction theory, is

$$\chi(\mathcal{O}_X, \mathcal{O}_X) - \chi(\mathcal{I}, \mathcal{I}),$$

where

$$\chi(A, B) = \sum_{i=0}^{3} (-1)^i \dim \mathrm{Ext}^i(A, B).$$

Since X is a nonsingular 3-fold, there exists a finite resolution of $\mathcal{I}$ by locally free sheaves,

$$0 \to F_3 \to F_2 \to F_1 \to F_0 \to \mathcal{I} \to 0.$$

Let x_{ij} denote the Chern roots of F_i. Since the determinant of $\mathcal{I}$ is trivial,

$$\sum_{i=0}^{3} \sum_j (-1)^i x_{ij} = 0.$$

Since the fundamental class of Y is β,

$$-\mathrm{ch}_2(\mathcal{I}) = \mathrm{ch}_2(\mathcal{O}_Y) = \beta.$$

We will calculate the virtual dimension in terms of the Chern roots via GRR. The first term is,

$$\chi(\mathcal{O}_X, \mathcal{O}_X) = \int_X \mathrm{Td}(X). \tag{2}$$

Next,

$$-\chi(\mathcal{I}, \mathcal{I}) = -\int_X \big(\sum_{i=0}^{3}\sum_j (-1)^i e^{-x_{ij}}\big) \cdot \big(\sum_{\hat{i}=0}^{3}\sum_{\hat{j}} (-1)^{\hat{i}} e^{x_{\hat{i}\hat{j}}}\big) \cdot \mathrm{Td}(X).$$

Since the Chern root expression in the integrand is *even*, only the components in degrees 0 and 2 need be considered. The degree 0 component is equal to 1, the square of the rank of $\mathcal{I}$. The integral of the degree 0 component against $\mathrm{Td}(X)$ cancels the first term (2). The degree 2 component is

$$\sum_{i,\hat{i}=0}^{3}\sum_{j,\hat{j}} (-1)^{i+\hat{i}} \left(\frac{x_{ij}^2}{2} - x_{ij}x_{\hat{i}\hat{j}} + \frac{x_{\hat{i}\hat{j}}^2}{2} \right) = 2\mathrm{ch}_2(\mathcal{I}) - \sum_{i,\hat{i}=0}^{3}\sum_{j,\hat{j}} (-i)^{i+\hat{i}} x_{ij}x_{\hat{i}\hat{j}}.$$

The second term on the right equals the square of the determinant of $\mathcal{I}$ and hence vanishes. We conclude the virtual dimension equals

$$-\int_X 2\mathrm{ch}_2(\mathcal{I}) \cdot \mathrm{Td}(X) = \int_\beta c_1(X)$$

since the degree 1 term of $\mathrm{Td}(X)$ is $c_1(X)/2$. □

The moduli space $I_n(X, \beta)$ is canonically isomorphic to the Hilbert scheme [15]. As the Hilbert scheme is a fine moduli space, universal structures are well-defined. Let π_1 and π_2 denote the projections to the respective factors of $I_n(X, \beta) \times X$. Consider the universal ideal sheaf $\mathfrak{I}$,

$$\mathfrak{I} \to I_n(X, \beta) \times X.$$

Since $\mathfrak{I}$ is π_1-flat and X is nonsingular, a finite resolution of $\mathfrak{I}$ by locally free sheaves on $I_n(X, \beta) \times X$ exists. Hence, the Chern classes of $\mathfrak{I}$ are well-defined.

For $\gamma \in H^l(X,\mathbb{Z})$, let $\mathrm{ch}_{k+2}(\gamma)$ denote the following operation on the homology of $I_n(X,\beta)$:

$$\mathrm{ch}_{k+2}(\gamma) : H_*(I_n(X,\beta),\mathbb{Q}) \to H_{*-2k+2-l}(I_n(X,\beta),\mathbb{Q}),$$

$$\mathrm{ch}_{k+2}(\gamma)\big(\xi\big) = \pi_{1*}\big(\mathrm{ch}_{k+2}(\mathfrak{I}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)\big).$$

We define descendent fields in Donaldson-Thomas theory, denoted by $\tilde{\tau}_k(\gamma)$, to correspond to the operations $(-1)^{k+1}\mathrm{ch}_{k+2}(\gamma)$. The descendent invariants are defined by

$$\langle \tilde{\tau}_{k_1}(\gamma_{l_1}) \cdots \tilde{\tau}_{k_r}(\gamma_{l_r}) \rangle_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} \prod_{i=1}^r (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_{l_i}),$$

where the latter integral is the push-forward to a point of the class

$$(-1)^{k_1+1}\mathrm{ch}_{k_1+2}(\gamma_{l_1}) \ \circ \ \cdots \ \circ \ (-1)^{k_r+1}\mathrm{ch}_{k_r+2}(\gamma_{l_r})\Big([I_n(X,\beta)]^{vir}\Big).$$

A similar slant product construction can be found in the Donaldson theory of 4-manifolds. Since the Chern character contains denominators, the descendent invariants in Donaldson-Thomas theory are rational numbers.

Define the Donaldson-Thomas partition function with descendent insertions by

$$\mathsf{Z}_{DT}\Big(X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i})\Big)_\beta = \sum_{n\in\mathbb{Z}} \langle \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}) \rangle_{n,\beta} \ q^n. \tag{3}$$

An elementary argument shows the charge n in the sum (3) is bounded from below. As before, the descendent insertions in (3) should match the virtual dimension.

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$\mathsf{Z}'_{DT}\Big(X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i})\Big)_\beta = \frac{\mathsf{Z}_{DT}\Big(X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i})\Big)_\beta}{\mathsf{Z}_{DT}(X;q)_0}.$$

The degree 0 partition function is determined by a conjecture of [14]. Following the Calabi-Yau case, we conjecture the series Z'_{DT} to be a rational function of q.

Conjecture 1. The degree 0 Donaldson-Thomas partition function for a 3-fold X is determined by:

$$\mathsf{Z}_{DT}(X;q)_0 = M(-q)^{\int_X c_3(T_X \otimes K_X)},$$

where

$$M(q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n}$$

is the McMahon function.

Conjecture 2. The reduced series $\mathsf{Z}'_{DT}\big(X;q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i})\big)_\beta$ is a rational function of q.

2.3 Primary fields

The GW/DT correspondence is easiest to state for the *primary* fields $\tau_0(\gamma)$ and $\tilde{\tau}_0(\gamma)$.

Conjecture 3. After the change of variables $e^{iu} = -q$,

$$(-iu)^d \, \mathsf{Z}'_{GW}\left(X;u \mid \prod_{i=1}^r \tau_0(\gamma_{l_i})\right)_\beta = (-q)^{-d/2} \, \mathsf{Z}'_{DT}\left(X;q \mid \prod_{i=1}^r \tilde{\tau}_0(\gamma_{l_i})\right)_\beta,$$

where $d = \int_\beta c_1(T_X)$.

Conjecture 3 is consistent with the calculation of degenerate contributions in [18]. Let C be a nonsingular, genus g curve in X which rigidly intersects cycles dual to the classes $\gamma_{l_1}, \ldots \gamma_{l_r}$. The local Gromov-Witten series is determined in [18],

$$\mathsf{Z}'_{GW}\left(X;u \mid \prod_{i=1}^r \tau_0(\gamma_{l_i})\right)_{[C]} = \left(\frac{\sin(u/2)}{u/2}\right)^{2g-2+d} u^{2g-2},$$

The local Donaldson-Thomas series is then *predicted* by Conjecture 3,

$$\begin{aligned}
\mathsf{Z}'_{DT}\left(X;q \mid \prod_{i=1}^r \tilde{\tau}_0(\gamma_{l_i})\right)_{[C]} &= (-iu)^d(-q)^{d/2}\left(\frac{e^{iu/2}-e^{-iu/2}}{iu}\right)^{2g-2+d} u^{2g-2} \\
&= q^{1-g}(1+q)^{2g-2+d}
\end{aligned}$$

The normalizations and signs in Conjecture 3 are fixed by the requirement that the reduced partition function Z'_{DT} has initial term q^{1-g} corresponding to the ideal of C.

If the cohomology classes γ_i are integral, the Donaldson-Thomas invariants for primary fields are *integer* valued. The integrality constraints for Gromov-Witten theory obtained via the GW/DT correspondence for primary fields were conjectured previously in [18, 19].

2.4 Descendent fields

For fixed curve class β, consider the full set of (normalized) reduced partition functions,

$$\mathsf{Z}'_{GW,\beta} = \left\{ (-iu)^{d-\sum k_i}\ \mathsf{Z}'_{GW}\left(X; u \mid \prod \tau_{k_i}(\gamma_{l_i})\right)_\beta \right\},$$

where $d = \int_\beta c_1(T_X)$ as before. Here, $\mathsf{Z}'_{GW,\beta}$ consists of the *finite* set of descendent series with insertions of the correct dimension. The set $\mathsf{Z}'_{GW,\beta}$ is partially ordered by $\sum k_i$, the *descendent* partial ordering. Similarly, let

$$\mathsf{Z}'_{DT,\beta} = \left\{ (-q)^{-d/2}\ \mathsf{Z}'_{DT}\left(X; q \mid \prod \tilde{\tau}_{k_i}(\gamma_{l_i})\right)_\beta \right\}.$$

Conjecture 4. After the change of variables $e^{iu} = -q$,

(i) the sets of functions $\mathsf{Z}'_{GW,\beta}$ and $\mathsf{Z}'_{DT,\beta}$ have the *same* linear spans,

(ii) there exists a canonical matrix expressing the functions $\mathsf{Z}'_{GW,\beta}$ as linear combinations of the functions $\mathsf{Z}'_{DT,\beta}$:

  (a) the matrix coefficients depend only upon the classical cohomology of X and universal series,

  (b) the matrix is unipotent and upper-triangular with respect to the descendent partial ordering.

By Conjecture 4, each element of $\mathsf{Z}'_{GW,\beta}$ is a canonical linear combination,

$$(-iu)^{d-\sum k_i}\ \mathsf{Z}'_{GW}\left(\prod \tau_{k_i}(\gamma_{l_i})\right)_\beta = (-q)^{-d/2}\ \mathsf{Z}'_{DT}\left(\prod \tilde{\tau}_{k_i}(\gamma_{l_i})\right)_\beta + \ldots, \quad (4)$$

where the omitted terms are strictly lower in the partial ordering.

We do not yet have a complete formula for the canonical matrix of Conjecture 4. However, for the descendents of the point class $[P] \in H^6(X, \mathbb{Z})$, we can formulate a precise conjecture.

Conjecture $4'$**.** After the change of variables $e^{iu} = -q$,

$$(-iu)^{d-\sum k_j} \ \mathsf{Z}'_{GW}\left(\prod \tau_0(\gamma_{l_i}) \prod \tau_{k_j}(P)\right)_\beta = \\ (-q)^{-d/2} \ \mathsf{Z}'_{DT}\left(\prod \tilde{\tau}_0(\gamma_{l_i}) \prod \tilde{\tau}_{k_j}(P)\right)_\beta ,$$

if $\operatorname{codim}(\gamma_{l_i}) > 0$ for each i.

2.5 Reactions

We believe the upper-triangular matrix of Conjecture 4 is determined by two types of *reactions*:

$$\begin{aligned} \tau_a(\gamma_l) &\rightarrow A_a(\gamma_l) \ \tau_{a-1}(c_1(T_X) \cup \gamma_l) \\ \tau_a(\gamma_l)\tau_{a'}(\gamma_{l'}) &\rightarrow A_{a,a'}(\gamma_l, \gamma_{l'}) \ \tau_{a+a'-1}(\gamma_l \cup \gamma'_l) \end{aligned}$$

The linear combination (4) should be generated by applying the two reactions to the Gromov-Witten insertions

$$\prod \tau_{k_i}(\gamma_{l_i})$$

to exhaustion and then interpreting the output in Donaldson-Thomas theory. For example,

$$(-iu)^{d-k} \ \mathsf{Z}'_{GW}\left(\tau_k(\gamma_l)\right)_\beta = \\ (-q)^{-d/2} \sum_{j=0}^{k} \left(\prod_{i=1}^{j} A_{k-i+1}(c_1(T_X)^{i-1} \cup \gamma_l) \right) \ \mathsf{Z}'_{DT}\left(\tilde{\tau}_{k-j}(c_1(T_X)^j \cup \gamma_l)\right).$$

The resulting matrix will be upper-triangular with respect to the *reaction partial ordering*, a refinement of the descendent partial ordering. We further speculate that the *reaction amplitudes*,

$$A_a(\gamma_l), \ A_{a,a'}(\gamma_l, \gamma_{l'}),$$

are given by universal formulas depending only upon the classical cohomology of X (including possibly the Hodge decomposition). Conjectures 3, 4, and $4'$ are all *consequences* of the reaction view of the GW/DT correspondence for descendent fields.

2.6 An example

Let X be $\mathbf{P}^3$ and let β be the class $[L]$ of a line. A Gromov-Witten calculation using localization and known Hodge integral evaluations yields the following result,

$$\mathsf{Z}'_{GW}\Big(X; u \mid \tau_0(L)\tau_1(P)\Big)_{[L]} = \left(\frac{\sin(u/2)}{u/2}\right)\cos(u/2)u^{-2},$$

see [7, 6]. By Conjecture $4'$,

$$\begin{aligned}
\mathsf{Z}'_{DT}\Big(X; q \mid \tilde{\tau}_0(L)\tilde{\tau}_1(P)\Big)_{[L]} &= (-iu)^3(-q)^2\Big(\frac{\sin(u/2)}{u/2}\Big)\cos(u/2)u^{-2} \\
&= (-iu)^3(-q)^2\frac{e^{iu/2}-e^{-iu/2}}{iu}\frac{e^{iu/2}+e^{-iu/2}}{2}u^{-2} \\
&= \frac{1}{2}q(1-q^2)
\end{aligned}$$

The resulting Donaldson-Thomas series can be checked order by order in q via localization.

3 The GW/DT correspondence for relative theories

3.1 GW theory

Let X be a nonsingular, projective 3-fold and let $S \subset X$ be a nonsingular divisor. The Gromov-Witten theory of X relative to S has been defined in [4, 9, 10, 12]. Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class satisfying

$$\int_\beta [S] \geq 0.$$

Let $\vec{\mu}$ be an ordered partition,

$$\sum \mu_j = \int_\beta [S],$$

with positive parts. The moduli space $\overline{M}'_{g,n}(X/S, \beta, \vec{\mu})$ parameterizes stable relative maps with possibly disconnected domains and relative multiplicities determined by $\vec{\mu}$. As usual, the connected components of the domain are required to map nontrivially. The target of a relative map is allowed to be a k-step degeneration, $X[k]$, of X along S, see [12].

The relative conditions in the theory correspond to partitions *weighted* by the cohomology of S. Let $\delta_1, \ldots, \delta_{m_S}$ be a basis of $H^*(S, \mathbb{Q})$. A cohomology weighted partition η consists of an *unordered* set of pairs,

$$\{(\eta_1, \delta_{\ell_1}), \ldots, (\eta_s, \delta_{\ell_s})\},$$

where $\sum_j \eta_j$ is an *unordered* partition of $\int_\beta [S]$. The automorphism group, $\mathrm{Aut}(\eta)$, consists of permutation symmetries of η.

The *standard* order on the parts of η is

$$(\eta_i, \delta_{\ell_i}) > (\eta_{i'}, \delta_{\ell_{i'}})$$

if $\eta_i > \eta_{i'}$ or if $\eta_i = \eta_{i'}$ and $\ell_i > \ell_{i'}$. Let $\vec{\eta}$ denote the partition $(\eta_1, \ldots, \eta_s)$ obtained from the standard order.

Relative Gromov-Witten invariants are defined by integration against the virtual class of the moduli of maps. Let $\gamma_1, \ldots, \gamma_{m_X}$ be a basis of $H^*(X, \mathbb{Q})$, and let

$$\langle \tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r}) \mid \eta \rangle'_{g,\beta} = \frac{1}{|\mathrm{Aut}(\eta)|} \int_{[\overline{M}'_{g,r}(X/S,\beta,\vec{\eta})]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_{l_i}) \cup \prod_{j=1}^{s} \mathrm{ev}_j^*(\delta_{\ell_j}).$$

Here, the second evaluations,

$$\mathrm{ev}_j : \overline{M}'_{g,r}(X/S, \beta, \vec{\eta}) \to S.$$

are determined by the relative points.

The Gromov-Witten invariant is defined for *unordered* weighted partitions η. However, to fix the sign, the integrand on the right side requires an ordering. The ordering is corrected by the automorphism prefactor.

As before, we will require the associated Gromov-Witten partition function,

$$\mathsf{Z}'_{GW}\Big(X/S; u \mid \prod_{i=1}^{r} \tau_{k_i}(\gamma_{l_i})\Big)_{\beta,\eta} = \sum_{g\in\mathbb{Z}} \langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_{l_i}) \ |\eta\rangle'_{g,\beta} \ u^{2g-2}. \tag{5}$$

The definitions here parallel those of Section 2.1.

3.2 DT theory

3.2.1 Stable relative ideal sheaves

Relative Donaldson-Thomas theory is defined via integration over the moduli space of relative ideal sheaves. We outline J. Li's definition of the relative theory here [13]. A full foundational treatment has not yet been written.

Let X be a nonsingular, projective 3-fold and let $S \subset X$ be a nonsingular divisor. Let $\mathcal{I}$ be an ideal sheaf on X with associated subscheme Y. The ideal sheaf $\mathcal{I}$ is *relative* to S if the natural map,

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_S \rightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_S,$$

is injective. Relativity may be viewed as a transversality condition of Y with respect to S. In particular, the scheme theoretic intersection, $Y \cap S$, defines an element of the Hilbert scheme,

$$\mathrm{Hilb}(S, \int_\beta [S]),$$

of points of S.

Relativity is an *open* condition on ideal sheaves on X. A proper moduli space, $I_n(X/S, \beta)$, of relative ideal sheaves is constructed by considering stable ideal sheaves relative on the degenerations $X[k]$ of X.

Let $S_0, \ldots, S_k$ denote the canonical images of S in the degeneration $X[k]$. Here, $S_0, \ldots, S_{k-1}$ are the singular divisors, and S_k is the transform of the original relative divisor. An ideal sheaf on $X[k]$ is *predeformable* if, for every singular divisor $S_l \subset X[k]$, the induced map,

$$\mathcal{I} \otimes_{\mathcal{O}_{X[k]}} \mathcal{O}_{S_l} \rightarrow \mathcal{O}_{X[k]} \otimes_{\mathcal{O}_{X[n]}} \mathcal{O}_{S_l}$$

is injective.

Let $Y_0, \ldots, Y_k$ be the restrictions of Y to the components of $X[k]$ with Y_k and Y_{k+1} incident S_k. The predeformability condition at the singular divisor S_l can be restated in the following form: Y_l and Y_{l+1} are transverse to S_l with equal scheme theoretic intersections,

$$Y_l \cap S_l = Y_{l+1} \cap S_l \subset S_l.$$

Ideal sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ on the degenerations $X[k_1]$ and $X[k_2]$ are isomorphic if $k_1 = k_2$ and there exists an isomorphism of varieties

$$\sigma : X[k_1] \to X[k_2]$$

over X such that

$$\sigma^*\{\mathcal{I}_2 \to \mathcal{O}_{X[k_2]}\} \cong \{\mathcal{I}_1 \to \mathcal{O}_{X[k_1]}\},$$

where the isomorphism $\sigma^*\mathcal{O}_{X[k_2]} \cong \mathcal{O}_{X[k_1]}$ is the identity. The automorphism group, $\mathrm{Aut}(\mathcal{I})$, is the set of equivalences of $\mathcal{I}$ to itself. A predeformable ideal sheaf $\mathcal{I}$ on $X[k]$ relative to S_k is *stable* if $\mathrm{Aut}(\mathcal{I})$ is finite.

The moduli space, $I_n(X/S, \beta)$, parameterizes stable, predeformable, ideal sheaves $\mathcal{I}$ on degenerations $X[k]$ relative S_k satisfying

$$\chi(\mathcal{O}_Y) = n$$

and

$$\pi_*[Y] = \beta \in H_2(X, \mathbb{Z}),$$

where $\pi : X[k] \to X$ is the canonical stabilization map. The moduli space $I_n(X/S, \beta)$ is a complete, Deligne-Mumford stack equipped with a canonical perfect obstruction theory.

Relative Donaldson-Thomas theory is defined via integration against the associated virtual class. The primary and descendent fields are defined via the Chern characters of the universal ideal sheaf $\mathfrak{I}$ on the universal product stack following Section 2.2. The predeformability condition is expected to imply the existence of finite resolutions of $\mathfrak{I}$ by locally free sheaves. The relative conditions in the theory are defined via the canonical intersection map,

$$\epsilon : I_n(X/S, \beta) \to \mathrm{Hilb}(S, \int_\beta [S]),$$

to the Hilbert scheme of points.

3.2.2 The Nakajima basis

The cohomology of the Hilbert scheme of points of S has a canonical basis indexed by cohomology weighted partitions. The basis is obtained from the representation of the Heisenberg algebra on the cohomologies of the Hilbert schemes of points [8, 16].

Let η be a cohomology weighted partition with respect to the basis $\delta_1, \ldots, \delta_{m_S}$ of $H^*(S, \mathbb{Q})$. Following the notation of [16], let

$$C_\eta = \frac{1}{\mathfrak{z}(\eta)} \, P_{\delta_1}[\eta_1] \cdots P_{\delta_s}[\eta_s] \cdot \mathbf{1} \ \in H^*(\mathrm{Hilb}(S, |\eta|), \mathbb{Q}), \tag{6}$$

where

$$\mathfrak{z}(\eta) = \prod_i \eta_i \ |\mathrm{Aut}(\eta)| \ ,$$

and $|\eta| = \sum_j \eta_j$. In the presence of odd cohomology, the sign of C_η is fixed by placing the operator product (6) in standard order.

The *Nakajima basis* of the cohomology of $\mathrm{Hilb}(S, k)$ is the set,

$$\{C_\eta\}_{|\eta|=k} \, ,$$

see [16].

We assume the cohomology basis of S is self dual with respect to the Poincaré pairing. Then, to each weighted partition η, a dual partition $\eta^\vee$ is defined by taking the Poincaré duals of the cohomology weights. The Nakajima basis is orthogonal with respect to the Poincaré pairing on the cohomology of the Hilbert scheme,

$$\int_{\mathrm{Hilb}(S,k)} C_\eta \cup C_\nu = \frac{(-1)^{k-\ell(\eta)}}{\mathfrak{z}(\eta)} \ \delta_{\nu,\eta^\vee} \, , \tag{7}$$

see [5, 16].

3.2.3 Relative Donaldson-Thomas invariants

Relative Donaldson-Thomas invariants are defined via integration over the moduli spaces of stable relative sheaves. The relative and absolute moduli spaces have the same deformation theories on the open set of ideal sheaves on X relative to S. Hence, $I_n(X/S, \beta)$ has virtual dimension $\int_\beta c_1(X)$ by

Lemma 1. Though the open set may be *empty*, the conclusion is correct, and we leave the details to the reader.

The descendent invariants in relative Donaldson-Thomas theory are defined by

$$\langle \tilde{\tau}_{k_1}(\gamma_{l_1}) \cdots \tilde{\tau}_{k_r}(\gamma_{l_r}) \mid \eta \rangle_{n,\beta} = \int_{[I_n(X/S,\beta)]^{vir}} \prod_{i=1}^{r} (-1)^{k_i+1} \mathrm{ch}_{k_i+2}(\gamma_{l_i}) \cap \epsilon^*(C_\eta).$$

Define the associated partition function by

$$\mathsf{Z}_{DT}\Big(X/S; q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i})\Big)_{\beta,\eta} = \sum_{n \in \mathbb{Z}} \langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}) \mid \eta \rangle_{n,\beta}\ q^n. \tag{8}$$

As before the charge n in the sum (3) is bounded from below.

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$\mathsf{Z}'_{DT}\Big(X/S; q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i})\Big)_{\beta,\eta} = \frac{\mathsf{Z}_{DT}\Big(X/S; q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i})\Big)_{\beta,\eta}}{\mathsf{Z}_{DT}(X/S; q)_0}.$$

We conjecture a complete formula for degree 0 relative theory. Let $\Omega_X[S]$ denote the locally free sheaf of differential forms of X with logarithmic poles along S. Let

$$T_X[-S] = \Omega_X[S]^{\vee},$$

denote the dual sheaf of tangent fields with logarithmic zeros. Let

$$K_X[S] = \Lambda^3 \Omega_X[S]$$

denote the logarithmic canonical class.

Conjecture 1R. The degree 0 relative Donaldson-Thomas partition function for a 3-fold X is determined by:

$$\mathsf{Z}_{DT}(X/S; q)_0 = M(-q)^{\int_X c_3(T_X[-S] \otimes K_X[S])}.$$

If S is empty, Conjecture 1R specializes to Conjecture 1′ of [14]. A proof of Conjecture 1R in the toric case is presented in Section 4. As before, we conjecture the reduced series are rational functions of q.

Conjecture 2R. The reduced series $\mathsf{Z}'_{DT}\big(X/S; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i})\big)_{\beta,\eta}$ is a rational function of q.

3.3 Primary fields

We restrict our discussion of the relative GW/DT correspondence to the primary fields. A treatment of the descendent correspondence at the level of Section 2.4 is left to the reader. In particular, we do not know the precise formulas for the descendent correspondence.

Conjecture 3R. After the change of variables $e^{iu} = -q$,

$$(-iu)^{d+\ell(\eta)-|\eta|}\mathsf{Z}'_{GW}\left(X/S; u \mid \prod_{i=1}^r \tau_0(\gamma_{l_i})\right)_{\beta,\eta} =$$
$$(-q)^{-d/2}\ \mathsf{Z}'_{DT}\left(X/S; q \mid \prod_{i=1}^r \tilde{\tau}_0(\gamma_{l_i})\right)_{\beta,\eta},$$

where $d = \int_\beta c_1(T_X)$ and $\ell(\eta)$ denotes the length of η.

We present the simplest example in which all the features of the correspondence are visible. Let D be a nonsingular surface, and let

$$X = \mathbf{P}^1 \times D.$$

Let $0, \infty \in \mathbf{P}^1$ be two points in the base, and let D_0 and D_∞ be the associated fibers. Let

$$[\mathbf{P}^1] \in H_2(X, \mathbb{Z})$$

denote the class of the horizontal $\mathbf{P}^1$, and let $\beta = m[\mathbf{P}^1]$. We will consider the theories of X relative to the divisors D_0 and D_∞ in the curve class β.

Since there are two divisors, the boundary conditions of the relative theories are specified by two partitions η and ν weighted by the cohomology of D. The relative Gromov-Witten theory is particularly simple to compute. A direct calculation yields the answer,

$$\mathsf{Z}'_{GW}\left(X/S; u\right)_{\beta,\eta,\nu} = \frac{1}{\mathfrak{z}(\eta)} u^{-2\ell(\eta)}\ \delta_{\nu,\eta^\vee}.$$

Our correspondence *predicts* the associated Donaldson-Thomas series,

$$\begin{aligned} \mathsf{Z}'_{DT}(X/S;q)_{\beta,\eta,\nu} &= (-q)^{d/2}(-iu)^{d-2m+\ell(\eta)+\ell(\nu)}\frac{1}{\mathfrak{z}(\eta)}u^{-2\ell(\eta)}\ \delta_{\nu,\eta^\vee} \\ &= \frac{(-1)^{m-\ell(\eta)}}{\mathfrak{z}(\eta)}q^m\ \delta_{\nu,\eta^\vee}, \end{aligned}$$

using the relation $d = 2m$ in the last equality.

The moduli space $I_m(X/D_0 \cup D_\infty, \beta)$ is isomorphic to $\mathrm{Hilb}(D, m)$. The Donaldson-Thomas invariant is therefore a classical intersection product,

$$\langle \eta|\ |\nu\rangle_{m,\beta} = \int_{\mathrm{Hilb}(D,m)} C_\eta \cup C_\nu.$$

The q^m term of the predicted Donaldson-Thomas series is thus correct by (7). The division of the degree 0 series does *not* affect the first term.

3.4 The degeneration formula

The relative theories satisfy degeneration formulas. Let

$$\lambda : \mathcal{X} \to C$$

be a nonsingular 4-fold fibered over a nonsingular, irreducible curve. Let X be a nonsingular fiber of λ, and let

$$X_1 \cup_S X_2$$

be a reducible special fiber consisting of two nonsingular 3-folds intersecting transversely along a nonsingular surface S. The degeneration formulas express the absolute invariants of X via the relative invariants of X_1/S and X_2/S. We will show the degeneration formulas of the relative theories are compatible with the GW/DT correspondence for primary fields.

The degeneration formula for Gromov-Witten theory is naturally written in terms of the absolute and relative partition functions,

$$\mathsf{Z}'_{GW}\left(X|\prod_{i=1}^r \tau_0(\gamma_{l_i})\right)_\beta =$$
$$\sum \mathsf{Z}'_{GW}\left(\frac{X_1}{S}|\prod_{i\in P_1}\tau_0(\gamma_{l_i})\right)_{\beta_1,\eta} \mathfrak{z}(\eta)u^{2\ell(\eta)}\ \mathsf{Z}'_{GW}\left(\frac{X_2}{S}|\prod_{i\in P_2}\tau_0(\gamma_{l_i})\right)_{\beta_2,\eta^\vee},$$

where the sum is over curve splittings $\beta_1 + \beta_2 = \beta$, marking partitions

$$P_1 \cup P_2 = \{1, \ldots, r\},$$

and cohomology weighted partitions η. The central factor on the right accounts for the multiplicities and the shift in the genus variable u. A proof can be found in [4, 9, 10, 12].

The degeneration formula for Donaldson-Thomas theory takes a very similar form,

$$\mathsf{Z}'_{DT}\left(X | \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{l_i})\right)_{\beta} =$$

$$\sum \mathsf{Z}'_{DT}\left(\frac{X_1}{S} | \prod_{i \in P_1} \tilde{\tau}_0(\gamma_{l_i})\right)_{\beta_1,\eta} \frac{(-1)^{|\eta|-\ell(\eta)}\mathfrak{z}(\eta)}{q^{|\eta|}} \mathsf{Z}'_{DT}\left(\frac{X_2}{S} | \prod_{i \in P_2} \tilde{\tau}_0(\gamma_{l_i})\right)_{\beta_2,\eta^\vee},$$

where the sum is as before. The central factor on the right accounts for the diagonal splitting,

$$[\triangle] = \sum_{|\eta|=k} (-1)^{k-\ell(\eta)}\mathfrak{z}(\eta)\ C_\eta \otimes C_{\eta^\vee} \quad \in H^*(\mathrm{Hilb}(S,k) \times \mathrm{Hilb}(S,k), \mathbb{Q}),$$

and the shift in the charge variable q. The proof should follow [12] but has yet to be written.

The compatibility between the degeneration formulas and the GW/DT correspondence is straightforward. Let $d = \int_\beta c_1(X)$ as before, and let

$$d_i = \int_{\beta_1} c_1(X_i).$$

We have a partition of the total degree d,

$$\begin{aligned} d &= d_1 + d_2 - 2\int_{\beta_1} [S] \\ &= (d_1 - |\eta|) + (d_2 - |\eta^\vee|). \end{aligned}$$

Using the degree partition, the degeneration formulas for the relative theories are equivalent via the GW/DT correspondence.

4 The equivariant vertex measure

4.1 Summary

Let $\mathbf{T}$ be a 3-dimensional complex torus with coordinates t_i. Let $\mathbf{T}$ act on $\mathbf{A}^3$ with coordinates x_i by

$$(t_1, t_2, t_3) \cdot x_i = t_i x_i \,. \tag{9}$$

In these coordinates, the tangent representation at the origin $0 \in \mathbf{A}^3$ has character $t_1^{-1} + t_2^{-1} + t_3^{-1}$.

Let π be a 3-dimensional partition with three outgoing 2-dimensional partitions λ_1, λ_2, and λ_3. The equivariant vertex V_π arises in the localization formula for the Donaldson-Thomas theory of toric 3-folds [14].

The equivariant vertex determines a natural 3-parametric family of measures w on 3-dimensional partitions. The measure of π is defined by

$$\mathsf{w}(\pi) = \prod_{k \in \mathbb{Z}^3} (s, k)^{-\mathsf{v}_k} \ ,$$

where $s = (s_1, s_2, s_3)$ are parameters, $(\,\cdot\,,\,\cdot\,)$ denotes the standard inner product, and v_k is the coefficient of t^k in V_π.

Consider the generating series of the equivariant vertex measures of 3-dimensional partitions π with fixed outgoing 2-dimensional partitions,

$$\mathsf{W}(\lambda_1, \lambda_2, \lambda_3) = \sum_{\pi} \mathsf{w}(\pi) q^{|\pi|}.$$

Here $|\pi|$ is defined as the (signed) number of boxes obtained by formally removing the infinite outgoing cylinders [14].

Theorem 1. *For finite 3-dimensional partitions,*

$$\mathsf{W}(\emptyset, \emptyset, \emptyset) = M(-q)^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}}.$$

Our proof is independent of the conjectural GW/DT correspondence. However, relative Donaldson-Thomas theory plays an essential role.

4.2 Equivariant Donaldson-Thomas theory

Let the 1-dimensional torus $\mathbf{T}^1$ act on $\mathbf{P}^1$ with tangent weights $-s_1$ and s_1 at the fixed points 0 and ∞. Let the 2-dimensional torus $\mathbf{T}^2$ act on $\mathbb{C}^2$ with weights $-s_2$ and $-s_3$. The torus $\mathbf{T} = \mathbf{T}^1 \times \mathbf{T}^2$ acts on

$$X = \mathbf{P}^1 \times \mathbb{C}^2$$

preserving the divisor S over ∞. We will study the equivariant Donaldson-Thomas theory of X relative to S.

Since X is not projective, the non-equivariant theory is not well-defined. However, the $\mathbf{T}$-equivariant theory can be defined via the residue since the $\mathbf{T}$-fixed locus of $I_n(X/S, 0)$ is projective. Let $\mathsf{Z}^{\mathbf{T}}_{DT}(X/S; q)_0$ denote the degree 0 partition function for the equivariant relative theory.

A rational function $f \in \mathbb{Q}(s_1, s_2, s_3)$ has only *monomial poles* in the variables s_2 and s_3 if

$$f(s_1, s_2, s_3) = \frac{p(s_1, s_2, s_3)}{s_2^{k_2} s_3^{k_3}}$$

for $p \in \mathbb{Q}[s_1, s_2, s_3]$ and $k_2, k_3 \in \mathbb{Z}$.

Lemma 2. *The q coefficients of $\mathsf{Z}^{\mathbf{T}}_{DT}(X/S; q)_0$ have only monomial poles in the variables s_2 and s_3.*

Proof. The Hilbert-Chow morphism and the collapsing maps,

$$X[k] \to X,$$

together yield a $\mathbf{T}$-equivariant, proper morphism,

$$\iota_1 : I_n(X/S, 0) \to \mathrm{Sym}^n(X).$$

The projection $X \to \mathbb{C}^2$ yields a $\mathbf{T}$-equivariant, proper morphism,

$$\iota_2 : \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(\mathbb{C}^2).$$

Finally, a $\mathbf{T}$-equivariant, proper morphism,

$$\iota_3 : \mathrm{Sym}^n(\mathbb{C}^2) \to \oplus_1^n \mathbb{C}^2,$$

is obtain via the higher moments,

$$\iota_3\Big(\ \{(x_i, y_i)\}\ \Big) = \Big(\sum_i x_i, \sum_i y_i\Big) \oplus \Big(\sum_i x_i^2, \sum_i y_i^2\Big) \oplus \cdots \oplus \Big(\sum_i x_i^n, \sum_i y_i^n\Big).$$

Let $j = \iota_3 \circ \iota_2 \circ \iota_1$.

The virtual class $[I_n(X/S,0)]^{vir}$ is an element of the $\mathbf{T}$-equivariant Chow ring of $I_n(X/S,0)$. Since j is $\mathbf{T}$-equivariant and proper,

$$\int_{[I_n(X/S,0)]^{vir}} 1 = \int_{\oplus_1^n \mathbb{C}^2} j_*[I_n(X/S,0)]^{vir},$$

where $\int$ denotes $\mathbf{T}$-equivariant integration. Since the space $\oplus_1^n \mathbb{C}^2$ has a unique $\mathbf{T}$-fixed point with tangent weights,

$$-s_2, -s_3, -2s_2, -2s_3, \ldots, -ns_2, -ns_3,$$

we conclude the integral

$$\int_{\oplus_1^n \mathbb{C}^2} j_*[I_n(X/S,0)]^{vir},$$

has only monomial poles in the variables s_2 and s_3. □

4.3 Localization

The $\mathbf{T}$-fixed loci of $I_n(X/S,0)$ lie over either 0 or ∞. The fixed points over 0 correspond to finite 3-dimensional partitions with localization contributions to $\mathsf{Z}^{\mathbf{T}}_{DT}(X/S;q)_0$ determined by $\mathsf{W}(\emptyset,\emptyset,\emptyset)$, see [14].

A Donaldson-Thomas theory of *rubber* naturally arises on the fixed loci of $I_n(X/S,0)$ over ∞. Let

$$R = \mathbf{P}^1 \times \mathbb{C}^2,$$

and let S_0 and S_∞ denote the divisors over 0 and ∞ respectively. Let $\mathbf{T}^2$ act on $\mathbb{C}^2$ with weights $-s_2$ and $-s_3$. We will consider the $\mathbf{T}^2$-equivariant Donaldson-Thomas rubber theory of R relative to S_0 and S_∞. For the rubber theory, sheaves differing by the $\mathbf{T}^1$ action on $\mathbf{P}^1$ are identified. We denote the rubber theory by a superscripted tilde.

The rubber moduli space $I_n(R/S_0 \cup S_\infty, 0)^{\sim}$ carries cotangent lines at the dynamical points 0 and ∞ of $\mathbf{P}^1$. Let ψ_0 denote the class of the cotangent line at 0. Let

$$\mathsf{W}_\infty = 1 + \sum_{n\geq 1} \int_{[I_n(R/S_0\cup S_\infty,0)^{\sim}]^{vir}} \frac{1}{s_1 - \psi_0},$$

where $\int$ here denotes $\mathbf{T}^2$-equivariant integration. The leading term 1 may be viewed as a degenerate $n = 0$ contribution. By the virtual localization formula applied to the relative Donaldson-Thomas theory of X/S, the series W_∞ generates the localization contributions to $\mathsf{Z}^{\mathbf{T}}_{DT}(X/S;q)_0$ of the $\mathbf{T}$-fixed points over ∞.

The product of the localization contributions over 0 and ∞ yields the partition function,

$$\mathsf{Z}^{\mathbf{T}}_{DT}(X/S;q)_0 = \mathsf{W}(\emptyset,\emptyset,\emptyset)\cdot \mathsf{W}_\infty. \tag{10}$$

Consider the $\mathbf{T}^2$-equivariant rubber theory *without* any cotangent line insertions,

$$\mathsf{F}_\infty = \sum_{n\geq 0} q^n \int_{[I_n(R/S_0\cup S_\infty,0)^\sim]^{vir}} 1.$$

By definition,

$$\mathsf{W}(\emptyset,\emptyset,\emptyset),\ \mathsf{W}_\infty \in \mathbb{Q}(s_1,s_2,s_3)[[q]],$$

and

$$\mathsf{F}_\infty \in \mathbb{Q}(s_2,s_3)[[q]].$$

Lemma 3. $\log W_\infty = \frac{1}{s_1}\mathsf{F}_\infty$.

Proof. We first expand W_∞ by powers of the cotangent line,

$$\mathsf{W}_\infty = 1 + \sum_{l\geq 0} \frac{1}{s_1^{l+1}}\mathsf{F}_{\infty,l}\,,$$

where

$$\mathsf{F}_{\infty,l} = \sum_{n\geq 1} q^n \int_{[I_n(R/S_0\cup S_\infty,0)^\sim]^{vir}} \psi_0^l.$$

Next, we apply a version of the topological recursion relation to inductively calculate $\mathsf{F}_{\infty,l}$. Let

$$\pi : \mathcal{Y}_n \to I_n(R/S_0\cup S_\infty,0)^\sim$$

be the universal subscheme over the moduli space. The morphism π is finite, flat, and compatible with the $\mathbf{T}^2$-action. Therefore,

$$q\frac{d}{dq}\mathsf{F}_{\infty,l} = \sum_{n>0} q^n \int_{[\mathcal{Y}_n]^{vir}} \psi_0^l,$$

where the virtual class of $\mathcal{Y}$ is defined as the pull-back of the virtual class of the moduli space by π. The canonical map,

$$f : \mathcal{Y}_n \to R[k],$$

projects further to $\mathbf{P}^1[k]$, the associated degeneration of the base $\mathbf{P}^1$. By the definition of the relative moduli space, the image in $\mathbf{P}^1[k]$ is always disjoint from the relative points 0 and ∞ and the nodes. Hence, the family of degenerating bases over $\mathcal{Y}_n$ has *three* disjoint nonsingular sections.

The application of the topological recursion relation determined by the three sections to ψ_0 yields the following equation,

$$q\frac{d}{dq}\mathsf{F}_{\infty,l} = \mathsf{F}_{\infty,l-1} \cdot q\frac{d}{dq}\mathsf{F}_{\infty,0}.$$

The solution,

$$\mathsf{F}_{\infty,l} = \frac{\mathsf{F}_{\infty,0}^{l+1}}{(l+1)!},$$

is easily found. We conclude $\mathsf{W}_\infty = \exp(\frac{1}{s_1}\mathsf{F}_\infty)$. □

4.4 Proof of Theorem 1

The logarithm of equation (10) yields the relation,

$$\log \mathsf{W}(\emptyset,\emptyset,\emptyset) = \log \mathsf{Z}^{\mathbf{T}}_{DT}(X/S;q) - \log \mathsf{W}_\infty.$$

By Lemmas 2 and 3, the q coefficients of $\log \mathsf{W}(\emptyset,\emptyset,\emptyset)$ are of the form

$$\frac{1}{s_1}\frac{p_1(s_1,s_2,s_3)}{p_2(s_2,s_3)},$$

where the p_i are polynomials. Since the equivariant vertex measure is a degree 0 rational function [14],

$$\deg(p_1) = 1 + \deg(p_2).$$

Since the series $\mathsf{W}(\emptyset,\emptyset,\emptyset)$ is *symmetric* in the variables s_1, s_2, and s_3, we conclude,

$$\log \mathsf{W}(\emptyset,\emptyset,\emptyset) = \frac{1}{s_1 s_2 s_3}\mathsf{F}_0(q,s_1,s_2,s_3),$$

where $\mathsf{F}_0 \in \mathbb{Q}[s_1, s_2, s_3][[q]]$. The coefficients of F_0 must be *cubic* polynomials.

By Lemma 4 below, the q^n coefficient of $\mathsf{W}(\emptyset, \emptyset, \emptyset)$ is divisible by the cubic factor $(s_1 + s_2)(s_1 + s_3)(s_2 + s_3)$ for all $n > 0$. Hence,

$$\log \mathsf{W}(\emptyset, \emptyset, \emptyset) = \frac{(s_1 + s_2)(s_1 + s_3)(s_2 + s_3)}{s_1 s_2 s_3} \overline{\mathsf{F}}_0(q).$$

The equivariant vertex measure is evaluated in the Calabi-Yau specialization in [14],

$$\log \mathsf{W}(\emptyset, \emptyset, \emptyset)|_{s_1+s_2+s_2=0} = M(-q).$$

Hence, $\overline{\mathsf{F}}_0 = -M(-q)$. $\square$

Lemma 4. *The q^n coefficient of $\mathsf{W}(\emptyset, \emptyset, \emptyset)$ is divisible by the cubic factor $(s_1 + s_2)(s_1 + s_3)(s_2 + s_3)$ for all $n > 0$.*

Proof. We will show the factor $s_1 + s_2$ occurs with positive multiplicity in the equivariant vertex measure $\mathsf{w}(\pi)$ for any finite plane partition π. By symmetry, the cyclic permutations of $s_1 + s_2$ also occur in $\mathsf{w}(\pi)$ with positive multiplicity.

Following the notation of [14], let $Q_\pi(t_1, t_2, t_3)$ be the characteristic polynomial of the partition π. Then, the character of the virtual tangent space at π is given by

$$\mathrm{V}_\pi(t_1, t_2, t_3) = Q_\pi - \frac{\bar{Q}_\pi}{t_1 t_2 t_3} + Q_\pi \bar{Q}_\pi \frac{(1 - t_1)(1 - t_2)(1 - t_3)}{t_1 t_2 t_3},$$

where $\bar{Q}_\pi(t_1, t_2, t_3) = Q_\pi(t_1^{-1}, t_2^{-1}, t_3^{-1})$. The vertex measure is obtained from V_π via the prescription

$$\sum \pm t_1^i t_2^j t_3^k \to \prod (i s_1 + j s_2 + k s_3)^{\mp 1}.$$

Hence, the monomials of the form $t_1^i t_2^i t_3^0$ in V_π are those which contribute a factor of $s_1 + s_2$ to $\mathsf{w}(\pi)$. The total multiplicity of $s_1 + s_2$ is the negative of the constant term in the Laurent polynomial $\mathrm{V}_\pi(x, x^{-1}, t_3)$.

Let ρ be a 2-dimensional partition. The *content* of the box (r, s) in ρ is $r - s$. The slices of π perpendicular to the z direction determine 2-dimensional partitions

$$\pi_0, \pi_1, \pi_2, \ldots \ .$$

Let $a_{i,j}$ be the number of boxes in π_j with content i. For convenience, we set $a_{i,j} = 0$ for $j < 0$. By definition, $Q_\pi(x, x^{-1}, t_3) = \sum_{i,j} a_{i,j} x^i t_3^j$.

The constant term of $\mathrm{V}_\pi(x, x^{-1}, t_3)$ may be expressed in terms of the contents. Using

$$\mathrm{V}_\pi(x, x^{-1}, t_3) = Q_\pi(x, x^{-1}, t_3) - \frac{\bar{Q}_\pi(x, x^{-1}, t_3)}{t_3} + Q_\pi \bar{Q}_\pi (2 - x - \frac{1}{x})(\frac{1}{t_3} - 1),$$

we find the constant term equals

$$a_{0,0} + \sum_{i,j \in \mathbb{Z}} (2a_{i,j+1}a_{i,j} - a_{i,j+1}a_{i+1,j} - a_{i+1,j+1}a_{i,j}) - (2a_{i,j}a_{i,j} - a_{i,j}a_{i+1,j} - a_{i+1,j}a_{i,j}).$$

We rewrite the constant term in a factored form,

$$a_{0,0} + \sum_{i,j \in \mathbb{Z}} \Big((a_{i,j+1} - a_{i+1,j+1})(a_{i,j} - a_{i+1,j}) - (a_{i,j} - a_{i+1,j})^2 \Big)$$

which equals

$$a_{0,0} - \frac{1}{2} \sum_{i,j \in \mathbb{Z}} \Big((a_{i,j} - a_{i+1,j}) - (a_{i,j+1} - a_{i+1,j+1}) \Big)^2. \tag{11}$$

Since $(a_{i,0} - a_{i+1,0}) = \pm 1$ or 0, we see

$$a_{0,0} = \sum_{i \geq 0} (a_{i,0} - a_{i+1,0}) = \sum_{i \geq 0} (a_{i,0} - a_{i+1,0})^2$$

with a similar equality for $i < 0$. Therefore, $a_{0,0}$ precisely cancels the $j = -1$ term in (11), yielding the expression

$$-\frac{1}{2} \sum_{i \in \mathbf{Z}, j \geq 0} ((a_{i,j} - a_{i+1,j}) - (a_{i,j+1} - a_{i+1,j+1}))^2 \tag{12}$$

for the constant term of $\mathrm{V}_\pi(x, x^{-1}, t_3)$.

We conclude (12) is negative since $a_{i,j} = 0$ for j sufficiently large. Hence, the multiplicity of $s_1 + s_2$ in $\mathsf{w}(\pi)$ is strictly positive. □

Corollary 1. *The degree 0 localization contributions over ∞ are:*

$$\mathsf{W}_\infty = M(-q)^{\frac{s_2 + s_3}{s_1}}.$$

Proof. By Lemmas 2 and 3, the Corollary is obtained by extracting the pole in s_1 of $\log \mathsf{W}(\emptyset, \emptyset, \emptyset)$. □

4.5 Degree 0 results for toric 3-folds

Let X be a nonsingular, projective, toric 3-fold equipped with a $\mathbf{T}$-action, and let $S \subset X$ be a nonsingular toric divisor.

Theorem 2. $Z_{DT}(X;q)_0 = M(-q)^{\int_X c_3(T_X \otimes K_X)}$.

Proof. Let $\{X_\alpha\}$ denote the set of $\mathbf{T}$-fixed points of X. By localization,

$$Z_{DT}(X;q)_0 = \prod_{X_\alpha} \mathsf{W}(\emptyset,\emptyset,\emptyset)|_{s_1=-s_1^\alpha,\ s_2=-s_2^\alpha,\ s_3=-s_3^\alpha},$$

where $s_1^\alpha, s_2^\alpha, s_3^\alpha$ are the tangent weights at X_α. By Theorem 1,

$$\log Z_{DT}(X;q)_0 = \Big(\sum_{X_\alpha} \frac{(-s_1^\alpha - s_2^\alpha)(-s_1^\alpha - s_3^\alpha)(-s_2^\alpha - s_3^\alpha)}{s_1^\alpha s_2^\alpha s_3^\alpha}\Big) \cdot \log M(-q).$$

The prefactor on the right is equal to $\int_X c_3(T_X \otimes K_X)$ by a direct application of the Bott residue formula. $\square$

Theorem 3. $Z_{DT}(X/S;q)_0 = M(-q)^{\int_X c_3(T_X[-S]\otimes K_X[S])}$.

Proof. Let $\{S_\gamma\}$ denote the set of $\mathbf{T}$-fixed points of S. Let s_1^γ be the normal weight to S at S_γ, and let s_2^γ, s_γ^3 be the tangent weights to S at S_γ. By localization,

$$\begin{aligned} Z_{DT}(X/S;q)_0 \ &= \prod_{X_\alpha \notin S} \mathsf{W}(\emptyset,\emptyset,\emptyset)|_{s_1=-s_1^\alpha,\ s_2=-s_2^\alpha,\ s_3=-s_3^\alpha} \ \cdot \\ &\quad \prod_{S_\gamma} \mathsf{W}_\infty|_{s_1=s_1^\gamma,\ s_2=-s_2^\gamma,\ s_3=-s_3^\gamma} . \end{aligned}$$

By Theorem 1 and Corollary 1,

$$\frac{\log Z_{DT}(X/S;q)_0}{\log M(-q)} = \sum_{X_\alpha \notin S} \frac{(-s_1^\alpha - s_2^\alpha)(-s_1^\alpha - s_3^\alpha)(-s_2^\alpha - s_3^\alpha)}{s_1^\alpha s_2^\alpha s_3^\alpha} + \sum_{S\gamma} \frac{-s_2^\gamma - s_3^\gamma}{s_1^\gamma}.$$

The weights of $T_X[-S] \otimes K_X[S]$ at S_γ are

$$-s_2^\gamma - s_3^\gamma, -s_2^\gamma, -s_3^\gamma.$$

Hence, the right side is equal to $\int_X c_3(T_X[-S] \otimes K_X[S])$ by the Bott residue formula. $\square$

4.6 Evaluations in higher degrees

While the equivariant vertex measure has a simple formula in degree 1,

$$\mathsf{W}(1, \emptyset, \emptyset) = (1+q)^{\frac{s_2+s_3}{s_1}} M(-q)^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}},$$

the higher degree cases are more subtle. We will study the evaluations in degrees 1 and higher in a future paper.

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